craft Trailing Vortices," AIAA Paper 72-42, Mountain View, Calif., 1972.

[19]Saffman, P. G., "The Motion of a Vortex Pair in A Stratified Atmosphere," *Studies in Applied Mathematics*, Vol. LI, 1972, pp. 107–119.

[20]Tombach, I. H., "Transport of a Vortex Wake in a Stably Stratified Atmosphere," *Aircraft Wake Turbulence and Its Detection*, Edited by J. H. Olsen et al., Plenum Press, New York. 1971, pp. 41–56.

[21]Tulin, M. P. and Shwartz, J., "The Motion of Turbulent Vortex-Pairs in Homogeneous and Density Stratified Media," TR. 231-15, AD 723 184, 1971, Hydronautics, Inc., Laurel, Md.

[22]Tombach, I. H., "Transport and Stability of a Vortex Wake," Final Rep. MRI 72-FR-1010, 1972, Meteorology Research, Inc., Altadena, Calif.

NOVEMBER 1973 J. AIRCRAFT VOL. 10, NO. 11

On the Inviscid Rolled-Up Structure of Lift-Generated Vortices

Vernon J. Rossow*
NASA Ames Research Center, Moffett Field, Calif.

A simple form is presented of the relationships derived by Betz for the inviscid, fully developed structure of lift-generated vortices behind aircraft. An extension is then made to arbitrary span-load distributions by inferring guidelines for the selection of rollup centers for the vortex sheet. These techniques are easier to use and yield more realistic estimates of the rolled-up structure of vortices than the original form of Betz' theory when the span loading differs appreciably from elliptic loading.

Introduction

LIFT-GENERATED vortices behind aircraft usually consist of two adjacent, well organized, oppositely rotating flow fields which are potentially dangerous to following aircraft. It is therefore highly desirable to be able to predict the structure of such a vortex pair for a wide variety of lift configurations in order to better assess the potential hazard and to explore ways to alleviate it. A theory by A. Betz[1] uses the three conservation equations for vortex systems to relate the structure of the vortex sheet behind an isolated wingtip (isolated half span) to the structure of a single, fully-developed vortex. Although this theory does not appear to have been used extensively in the past, it has recently been demonstrated by Donaldson[2] to be useful and often more accurate than more complex methods. The favorable publicity given to Betz' method by Donaldson led to an elaboration of the theory and more examples by Mason and Marchman[3] and to the use by Brown[4] to predict the axial flow velocity in the vortex. These papers used the rollup equations in about the same form presented by Betz.

This paper presents a new form of the rollup relationships that is simpler in form and easier to use. These equations are then applied to several span-load distributions to illustrate the variations in vortex structure produced by various span-load distributions. After the method is generalized to include situations wherein the vortex sheet rolls up into several vortices on each side of the fuselage, a span-load distribution typical of current large aircraft is analyzed.

Derivation of Simplified Form of Rollup Equations

The three-dimensional shape of the vortex sheet as it rolls up behind a lifting wing is often approximated by considering the sheet at its intersection with the Trefftz plane, which is a plane behind the wing perpendicular to the freestream (see Fig. 1). The Trefftz plane approximation makes it possible to treat the motion of the vortex sheet as a two-dimensional, time-dependent calculation without axial flow. The Betz method does not treat the transition or intermediate stages between the initial vortex sheet behind the wing and the final rolled-up vortex structure. It simply uses the three conservation relations for two-dimensional vortex systems to relate the span-load distribution to the fully-developed vortex structure. In order to achieve a unique result, Betz assumed that the rollup process is independent of any other vortices that may form in the wake and that it is orderly so that the vorticity shed at the wingtip goes into the center of the vortex located at the spanwise centroid of vorticity. Each inboard portion of the sheet then forms a layer of vorticity around all of the previous wrappings until the entire sheet is rolled around the original center, as indicated in Fig. 1.

The spanwise variation of lift on the wing, $l(y)$, is taken to be

$$l(y) = \rho U_\infty \Gamma_w(y) \tag{1}$$

where ρ is the air density, U_∞ the freestream velocity, and $\Gamma_w(y)$ the span-wise variation of circulation or bound vorticity on the wing. The three conservation laws that re-

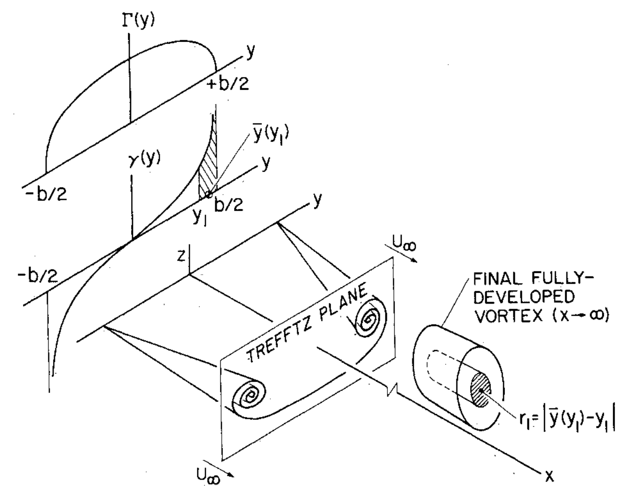

Fig. 1. Schematic diagram of relationship between span loading, $\Gamma_w(y)$, vortex sheet, $\gamma_w(y)$, Trefftz plane, and final rolled-up vortex for one side.

Received March 30, 1973; revision received August 10, 1973.
Index categories: Aircraft Aerodynamics.

*Staff Scientist. Associate Fellow AIAA.

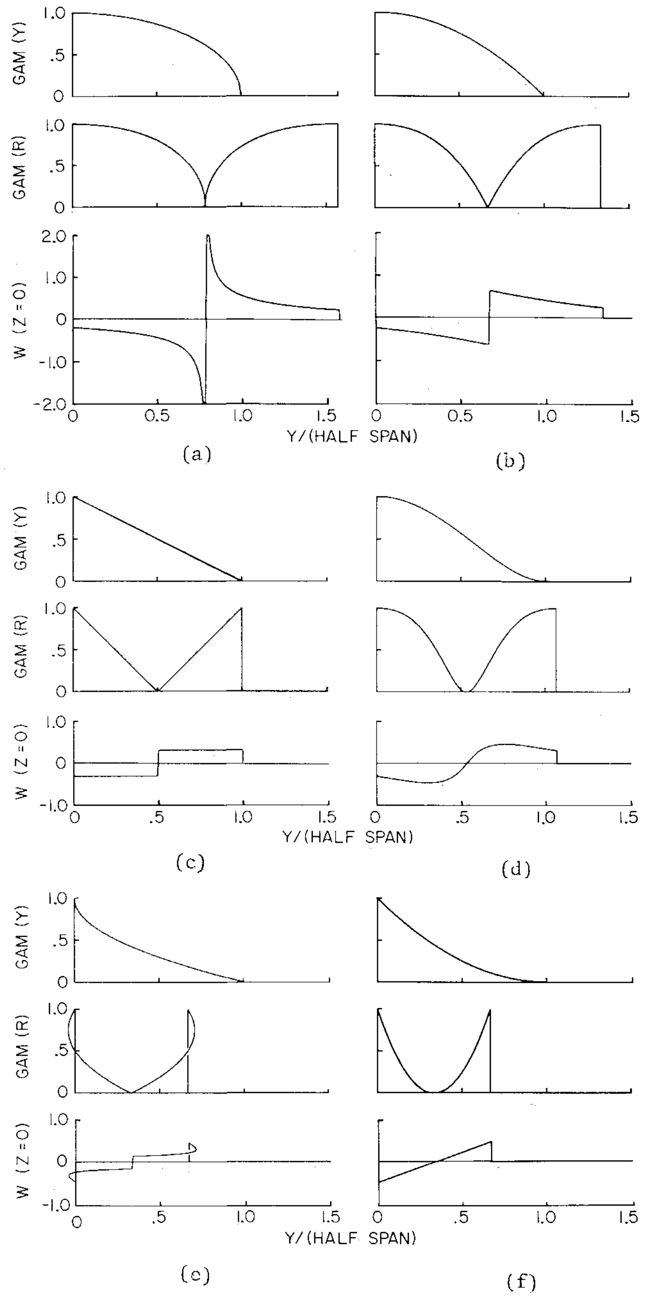

Fig. 2. Structure of fully developed vortex assuming that rollup begins at the wingtip end of the vortex sheet (Eq. (12)) for various span loadings represented by $\Gamma_w(y)/\Gamma_0 = [1 - (2y/b)^N]^M$. a) $N = 2.0$, $M = 0.5$; b) $N = 2.0$, $M = 1.0$; c) $N = 1.0$, $M = 1.0$; d) $N = 2.0$, $M = 2.0$; e) $N = 0.5$, $M = 1.0$; f) $N = 1.0$, $M = 2.0$.

late the circulation on the wing to that in the fully developed vortex $\Gamma_v(r)$ are then:

I. The circulation is conserved

$$\Gamma_w(0) = \Gamma_0 = -\int_0^{b/2} \frac{d\Gamma_w(y)}{dy} dy = \int_0^{r_{max}} \frac{d\Gamma_v(r)}{dr} dr \quad (2)$$

II. The centroid of vorticity remains at a fixed spanwise location

$$\bar{y}(0) = \frac{1}{\Gamma_0} \int_{b/2}^{0} y \frac{d\Gamma_w(y)}{dy} dy \quad (3)$$

III. The second moment of vorticity is conserved; $J_v = J_w = J$

$$J = \int_{b/2}^{0} [\bar{y}(0) - y]^2 \frac{d\Gamma_w(y)}{dy} dy = \int_0^{r_{max}} r^2 \frac{d\Gamma_v(r)}{dr} dr \quad (4)$$

where r_{max} is the radius within which all of the circulation is contained in the fully-developed vortex.

Equations (2, 3, and 4) are next assumed by Betz to apply piecewise, beginning at the wingtip, to successive portions of the sheet in toward the wing root. These segments of vortex sheet are assumed to be wrapped in the same sequence from the span loading onto the center of the vortex out to r_{max}. The equations that relate the vorticity in the sheet to that in the vortex may then be written as

$$\int_{b/2}^{y_1} \frac{d\Gamma_w(y)}{dy} dy = \int_0^{r_1} \frac{d\Gamma_v(r)}{dr} dr \quad (5)$$

$$\bar{y}(y_1) = \frac{1}{\Gamma_w(y_1)} \int_{b/2}^{y_1} y \frac{d\Gamma_w(y)}{dy} dy \quad (6)$$

$$\int_{b/2}^{y_1} [\bar{y}(y_1) - y]^2 [d\Gamma_w(y)/dy] dy = \int_0^{r_1} r^2 [d\Gamma_v(r)/dr] dr \quad (7)$$

where the relationship between the two independent variables, r_1 and y_1, in the rollup process is yet to be determined. Since the rollup is assumed to progress continuously from the wingtip inboard to the wing root, Eqs. (5) and (7) can be applied piecewise so that

$$-[d\Gamma_w(y_1)/dy_1]dy_1 = [d\Gamma_v(r_1)/dr_1]dr_1 \quad (8)$$

and

$$-[\bar{y}(y_1) - y_1]^2 [d\Gamma_w(y_1)/dy_1] dy_1 = r_1^2 [d\Gamma_v(r_1)/dr_1] dr_1 \quad (9)$$

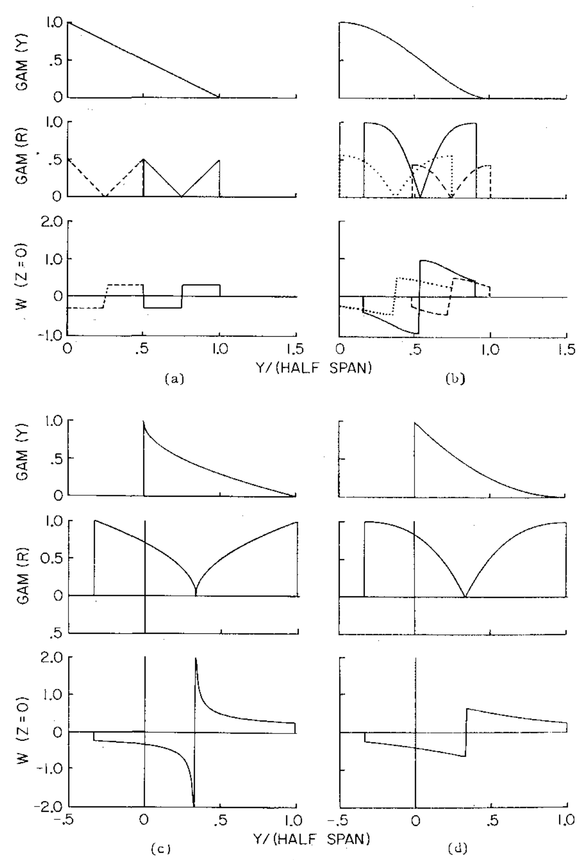

Fig. 3. Structure of fully developed vortices for the same span loadings presenzd in Figs. 2c-2f but with rollup beginning points determined by rules inferred in this paper and then using Eq. (17). a) $N = 1.0$, $M = 1.0$; b) $N = 2.0$, $M = 2.0$; c) $N = 0.5$, $M = 1.0$; d) $N = 1.0$, $M = 2.0$.

These two relationships can be combined to yield

$$r_1 = |\bar{y}(y_1) - y_1| \tag{10}$$

so that, $r_{max} = \bar{y}(0)$. Equation (6) for $\bar{y}(y_1)$ can be simplified further by integrating by parts to yield

$$\bar{y}(y_1) = y_1 - \frac{1}{\Gamma_w(y_1)}\int_{b/2}^{y_1} \Gamma_w(y)dy \tag{11}$$

or,

$$r_1 = \left|\frac{1}{\Gamma_w(y_1)}\int_{b/2}^{y_1} \Gamma_w(y)dy\right| \tag{12}$$

For an axially symmetric vortex, the circumferential velocity is

$$v_\theta = \Gamma_v(r_1)/2\pi r_1 \tag{13}$$

The foregoing simplified form of the Betz rollup equations permits the evaluation in closed form of the vortex structures for a variety of span-load distributions. For example, the radial variation of circulation and velocity in that part of the vortex that contains vorticity becomes for elliptic span loading

$$\frac{\Gamma_w(y_1)}{\Gamma_0} = \left[1 - \left(\frac{2y_1}{b}\right)^2\right]^{1/2}$$

$$\frac{2r_1}{b} = \frac{\pi/4 - 1/2 \sin^{-1}2y_1/b}{\left[1 - \left(\frac{2y_1}{b}\right)^2\right]^{1/2}} - \frac{y_1}{b}$$

and for parabolic span loading the results are

$$\frac{\Gamma_w(y_1)}{\Gamma_0} = 1 - \left(\frac{2y_1}{b}\right)^2$$

$$\frac{2r_1}{b} = \frac{2}{3}\,\frac{1 + 2y_1/b + (2y_1/b)^2}{(1 + 2y_1/b)} - \frac{2y_1}{b}$$

which agrees with the result presented by Brown.[4] The expressions for triangular span loading are

$$\frac{\Gamma_w(y_1)}{\Gamma_0} = 1 - \left|\frac{2y_1}{b}\right|$$

$$\frac{2r_1}{b} = \left(1 - \left|\frac{2y_1}{b}\right|\right)/2$$

so,

$$v_\theta = 2\Gamma_0/\pi b, \qquad r_1 \leq b/4$$

$$= \Gamma_0/2\pi r_1, \qquad r_1 > b/4$$

To gain an understanding of how the structure of the rolled-up vortex changes with variations in span-load distribution, a series of cases were calculated using Eqs. (12) and (13). The function used for the span load is

$$\Gamma_w(y)/\Gamma_0 = GAM(Y) = (1 - Y^N)^M \tag{14}$$

where $Y = 2y/b$, and $W(z = 0) = bw)_{z=0}/2\Gamma_0$. The $z = 0$ plane is defined as the horizontal plane through the center of the developed vortex. Figure 2 presents the spanwise loading, circulation, and vertical velocity for the rolled-up vortex for several values of N and M, assuming that rollup begins at the wingtip as postulated in the original Betz theory. Instead of plotting $\Gamma_v(r)/\Gamma_0$ and v_θ as a function of radius $R = 2r/b$, the results are presented at their position on the $Y = 2y/b$ axis to better illustrate their spanwise location. The curves for velocity in these figures, and in the ones to follow, terminate at the edges of the region where vorticity is located and so do not present the velocity in the irrotational part of the flow field. The double-valued curve in Fig. 2e is a consequence of choosing the improper rollup center.

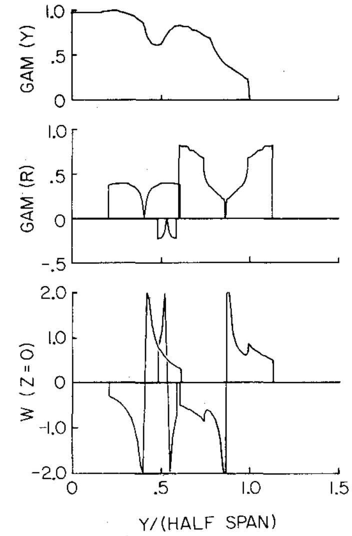

Fig. 4. Structure of multiple vortices that are estimated to form behind span loading typical of current large aircraft in landing configuration; $C_L = 1.5$.

Extension to Arbitrary Span Loadings

The double-valued character of the circulation and of the velocity in the developed vortex shown in Fig. 2e suggests that the vortex sheet should be subdivided differently before applying the Betz theory. An exact evaluation of the starting points for rollup would require a calculation of the complete time history of the change of the vortex sheet into the final individual vortices. Since this is difficult to do and an approximate estimate would suffice for many purposes, the rollup locations are related here to the down-wash velocity at the wing trailing edge. This vertical velocity is given by

$$w(y) = +\frac{1}{4\pi}\int_{-b/2}^{+b/2} \frac{\gamma_w(\eta)d\eta}{y - \eta} \tag{15}$$

Since the integral cannot be evaluated in the closed form for the general case, a relationship between $w(y)$ and $\gamma_w(y)\{= -[d\Gamma_w(y)/dy]\}$ can be obtained by repeatedly integrating Eq. (15) by parts to form the series

$$w(y) = -\frac{1}{4\pi}\left\{\gamma_w(y)\ln\left(\frac{b/2 - y}{b/2 + y}\right) + \sum_{n=1}^{\infty}\frac{(b/2 - y)^n - (-b/2 - y)^n}{n \cdot n!}\frac{d^n\gamma_w(y)}{dy^n}\right\} \tag{16}$$

An estimate for the beginning points for rollup is taken as those places where the vertical velocity of the sheet $w(y)$ changes sign or changes abruptly; or, by Eq. (16), at those places on the sheet where $d\gamma_w(y)/dy$ is at maximum or changes abruptly. For example, $d\gamma_w(y)/dy$ is infinite at the wingtip for elliptic loading (γ_w is also infinite there). For the triangularly loaded case, the derivative $d\gamma_w(y)/dy$ is infinite at both wingtips and at midspan ($y = 0$) because $\gamma_w(y)$ is discontinuous there. In Figs. 2e and 2f, the strength of the inboard or wing root portion of the vortex sheet is a maximum and the vortex strength would, in fact, be discontinuous across the junction of the left and right wing, so that rollup is estimated to begin at center span rather than at the wingtip in both of those cases. The parabolic and contoured loading shown in Figs. 2b and 2d have their maximum values of γ_w, respectively, at the tip and about halfway out (i.e., at $y/_{b/2} = 0.4444$). As already mentioned for parabolic loading, the wingtip is chosen as the rollup site because $\gamma_w(y)$ is discontinuous there, being finite for $y \leq b/2$ and zero for $y > b/2$.

If two or more vortices are produced on each side of the wing, the vortex sheet is divided into rollup segments at those places where $\gamma_w(y)$ or its derivative is zero. In triangular loading, the sheet is divided at $y = b/4$, so that the two vortices are of equal strength.

In order to perform the calculations for vortices originating at several places along the semispan, Eq. (12) is rewritten so that rollup begins at the estimated correct rollup site, $y = y_B$

$$r_1 = \left| \frac{1}{\Gamma_w(y_1) - \Gamma_w(y_B)} \int_{y_B}^{y_1} [\Gamma_w(y) - \Gamma_w(y_B)] dy \right| \quad (17)$$

If the rollup site occurs at midpoint in the sheet, Eq. (17) is applied to the two segments of the sheet separately, and the two resulting curves for $\Gamma_v(r_1)$ are added. That is, it is assumed that the two parts of the sheet roll up within one another without interacting. The final variation of $\Gamma_v(r_1)[= \Gamma_{\text{left}}(r_1) + \Gamma_{\text{right}}(r_1)]$ is then used in Eq. (13) for the calculation of the velocity. The center of the vortex is, of course, located at the centroid of the segment. Figure 3 shows the application of these techniques to the four cases that do not have the entire rollup beginning at the wingtip.

A calculation using the foregoing rules for the vortices to be expected far behind a current large aircraft with flaps deflected is shown in Fig. 4. Since these calculations are made as if each vortex were isolated from all the others, overlap of the various vorticity distributions occurs. An approximate correction for overlap of the individual vortices (that was not used here) is to superimpose all of the individual vortex fields so that nonaxially symmetric streamlines are obtained. The streamlines in their new locations each possess the same vorticity as before superposition, so that the centroids are no longer necessarily located at the centers estimated for the vortices. Such a superposition usually ignores this shift and any relative motion of the vortices that occurs as they orbit about one another during their development and after they are fully formed. The resulting flow field would then be applicable at the instant of time when the vortices are alined as assumed in the superposition. It is felt therefore that the effort required to superimpose the vortex flow fields and to calculate new streamline paths is not warranted.

Conclusions

The simplified form of the Betz rollup equations derived and the extensions suggested in this paper make it possible to estimate easily the structure of vortices that trail behind wings with arbitrary span-load distributions. The rules inferred for subdividing the vortex sheet into separate segments and for identifying the beginning points of rollup for each segment can be summarized as follows:

1. Vortex rollup sites are located at maxima of sheet strength and at abrupt changes in sheet strength.
2. The edges of the segment of vortex sheet that rolls into a vortex occur where the sheet strength vanishes or changes sign, or where the sheet strength is at minimum.

The improved theory is still approximate in that the interaction of the vortices with one another is ignored, along with viscosity and variations in the axial velocity. Also, it is assumed that vorticity from the sheet enters the vortex in sequence from its position relative to its neighbors in the sheet, so that orbiting or interchanging of positions of elements of the sheet is ruled out.

References

[1]Betz, A., "Verhalten von Wirbelsystemen," *Zeitschrift fuer Angewandte Mathematik und Mechanik*, Bd. XII, Nr. 3, 1932, pp. 164–174; see also TM 713, NACA.

[2]Donaldson, C. duP., "A Brief Review of the Aircraft Trailing Vortex Problem," ARAP Rept. 155, May 1971, Aeronautical Research Associates of Princeton, Princeton, N.J.

[3]Mason, W. H. and Marchman, J. F., III, "Farfield Structure of an Aircraft Trailing Vortex, Including Effects of Mass Injection," CR-62078, April 1972, NASA.

[4]Brown, C. E., "Aerodynamics of Wake Vortices," *AIAA Journal*, Vol. 11, No. 4, April 1973, pp. 531–536.